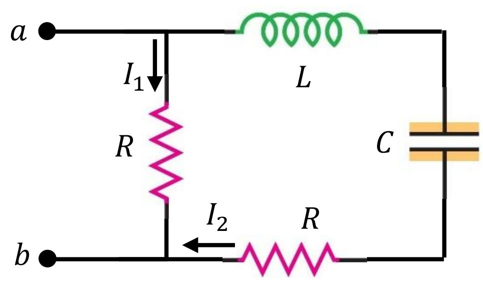
PHYS 102 – General Physics II Final Exam

Duration: 120 minutes Sunday, 24 December 2023; 18:00

1. Consider the circuit given.



(a) (10 Pts.) At time $t = 0$, a DC source is connected to the circuit so that the voltage across the points a and b is V. Find the currents in each resistor at time $t = 0$, and after a very long time.

(b) (10 Pts.) Instead of a DC source, an AC source $V(t) = V_0 \cos \omega t$ is connected at the points a and b for a very long time ($t \to \infty$). Find the time dependent currents in each resistor.

(c) (5 Pts.) What is the average power dissipated in the circuit when the system is in resonance?

Solution:

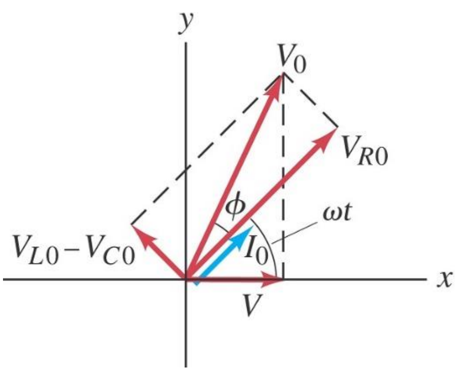
(a) At time $t = 0$, L behaves like an open circuit. Therefore we have $I_1 = V/R$, and $I_2 = 0$. In the limit $t \to \infty$, C is fully charged, and hence no current passes through the capacitor. Therefore, we again have $I_1 = V/R$, and $I_2 = 0$.

(b) The current I_1 is easily found using Ohm's law

$$I_1 = \frac{V_0}{R} \cos \omega t\,.$$

Phasor diagram for the RLC branch of the circuit is shown in the figure. Accordingly, we have

$$V_0 = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} = I_2\sqrt{R^2 + (X_L - X_C)^2}.$$



Thus, total, impedance Z, and the phase ϕ of the RLC branch is

$$Z = \frac{V_0}{I_2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\,, \qquad \phi = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

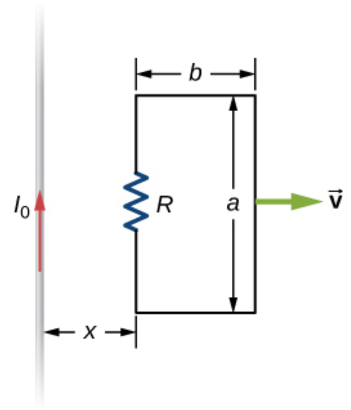
and,

$$I_2 = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\,, \qquad I = I_1 + I_2\,.$$

(c) When the system is in resonance, $\omega L = 1/\omega C \quad \to \quad \phi = 0$, and $I = \frac{2V_0}{R} \cos \omega t\,.$

$$\bar{P} = \frac{1}{2} V_0 I_0 = \frac{V_0^2}{R}\,.$$

2. A rectangular circuit containing a resistance R is pulled at a constant velocity $\vec{\mathbf{v}}$ away from a long, straight wire carrying a current I_0. (Figure on the right.)



(a) (5 Pts.) Is the current induced in the circuit clockwise or counter clockwise?

(b) (20 Pts.) Derive an equation that gives the current induced in the circuit as a function of the distance x between the near side of the circuit and the wire.

Solution:

(a) Magnetic field created by the infinite current at a point in the rectangular circuit is perpendicular to the plane of the rectangular circuit and inward. By the symmetry of the current, its magnitude only depends on the perpendicular distance to the current, and can be found using Ampère's law by considering a circle of radius r centered on the current.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I_0 \quad \rightarrow \quad B(r)(2\pi r) = \mu_0 I_0 \quad \rightarrow \quad B(x) = \frac{\mu_0 I_0}{2\pi r}.$$

Flux of this magnetic field through the rectangular circuit is found as

$$d\Phi = B\,dA = B(r)\,a\,dr \quad \rightarrow \quad \Phi = \frac{\mu_0 I_0 a}{2\pi}\int_x^{x+b}\frac{dr}{r} \quad \rightarrow \quad \Phi = \frac{\mu_0 I_0 a}{2\pi}\ln\left(1+\frac{b}{x}\right).$$

Since the flux decreases as x increases, by Lenz's law, the current is clockwise to oppose the decrease.

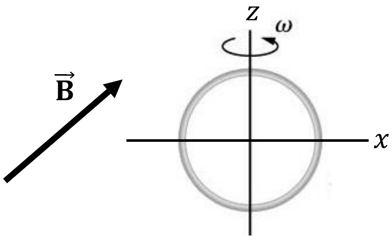
(b) By Faraday's law, absolute value of the electromotive force induced araund the rectangular circuit is

$$|\mathcal{E}| = \frac{d\Phi}{dt} \quad \rightarrow \quad |\mathcal{E}| = \frac{\mu_0 I_0 a}{2\pi}\left(\frac{x}{x+b}\right)\left(-\frac{b}{x^2}\right)\frac{dx}{dt} \quad \rightarrow \quad |\mathcal{E}| = \frac{\mu_0 I_0 ab\,v}{2\pi x(x+b)}.$$

Finally, we have

$$I_{\text{ind}} = \frac{|\mathcal{E}|}{R} = \frac{\mu_0 I_0 ab\,v}{2\pi R x(x+b)}.$$

3. A circular loop of wire of radius r is mounted on a vertical shaft oriented along the z axis and rotated at an angular frequency ω in a region of uniform magnetic field $\vec{\mathbf{B}} = B_x\hat{\mathbf{\imath}} + B_z\hat{\mathbf{k}}$.



(a) (10 Pts.) Find an expression for the time-dependent flux through the loop.

(b) (15 Pts.) Determine the magnitude of the time-dependent current through the loop if it has a resistance R and inductance L.

Solution:

(a) The circular loop rotates around the z axis with angular frequency ω therefore, its normal vector at any time is

$$\hat{\mathbf{n}} = \cos(\omega t + \phi)\ \hat{\mathbf{\imath}} + \sin(\omega t + \phi)\ \hat{\mathbf{\jmath}}$$

where the phase ϕ is determined by the position of the loop at time $t = 0$. The flux of the magnetic field through the loop is

$$\Phi = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}} = \vec{\boldsymbol{B}} \cdot \vec{\mathbf{n}}\, A \quad \rightarrow \quad \Phi = B_x \pi r^2 \cos(\omega t + \phi)\,.$$

(b) By Faraday's law, the emf induced around the loop is

$$|\mathcal{E}| = \frac{d\Phi}{dt} \quad \rightarrow \quad |\mathcal{E}| = -B_x \pi r^2 \omega \sin(\omega t + \phi)\,.$$

Magnitude of the time-dependent current through the loop is

$$|I| = \frac{|\mathcal{E}|}{Z}$$

where Z is the impedance of the circuit. Since for an LR circuit $Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2}$, we have

$$|I| = \frac{B_x \pi r^2 \omega}{\sqrt{R^2 + \omega^2 L^2}}\,.$$

4. The electric field of an electromagnetic wave has the following components:

$E_x = (9\ N/C)\cos(k\,z - 300\,\pi\,t)\,, \qquad E_y = (9\ N/C)\cos(k\,z - 300\,\pi\,t)\,, \qquad E_z = 0\,.$

Use $\mu_0 = 4\pi \times 10^{-7}$ T.m/A, and $c = 3 \times 10^8$ m/s for the speed of light to answer the following questions. Indicate the correct SI units for all numerical answers, and **DO NOT** use rounded value of π. Leave it as a symbol.

(a) (3 Pts.) What is the value of k? ($k > 0$)

(b) (3 Pts.) What is the wavelength of this electromagnetic wave?

(c) (3 Pts.) In which direction is the wave moving?

(d) (5 Pts.) Find the *x*, *y* and *z* components of the magnetic field of this wave.

(e) (5 Pts.) What is the maximum magnitude of the Poynting vector?

(f) (3 Pts.) What is the time averaged power passing through a surface area of 1 m^2 in the *xy* plane?

(g) (3 Pts.) What is the time averaged power passing through a surface area of 1 m^2 in the *xz* plane?

Solution: (a)

$$\omega = 300\,\pi = ck \quad \rightarrow \quad k = \frac{300\pi}{3 \times 10^8} = \pi \times 10^{-6}\ \mathrm{m}^{-1}$$

(b)

$$\lambda = \frac{2\pi}{k} \quad \rightarrow \quad \lambda = 2 \times 10^6\ \mathrm{m}$$

(c)

$\hat{\mathbf{k}}$ direction ($+z$ direction)

(d)

$$|\vec{\mathbf{B}}| = \frac{|\vec{\mathbf{E}}|}{c}\,, \qquad \vec{\mathbf{B}} \cdot \hat{\mathbf{k}} = 0 \quad \rightarrow \quad B_z = 0$$

$$\vec{\mathbf{B}} \cdot \vec{\mathbf{E}} = 0 \quad \rightarrow \quad B_x E_x + B_y E_y = 0 \quad \rightarrow \quad B_x \sim \pm E_y\,, \qquad B_y \sim \mp E_x$$

$$\vec{\boldsymbol{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}} = \frac{EB}{\mu_0}\hat{\mathbf{k}} \quad \rightarrow \quad E_x B_y - E_y B_x > 0$$

$$B_x = (-3 \times 10^{-8}\ \mathrm{T})\cos(k\,z - 300\,\pi\,t)\,, \qquad B_y = (3 \times 10^{-8}\ \mathrm{T})\cos(k\,z - 300\,\pi\,t)$$

(e)

$$|\vec{\boldsymbol{S}}| = \frac{1}{\mu_0}|\vec{\mathbf{E}} \times \vec{\mathbf{B}}| = \frac{EB}{\mu_0} = \frac{E^2}{\mu_0 c} = \frac{2(81)}{(4\pi \times 10^{-7})(3 \times 10^8)} = \frac{27}{2\pi} \times 10^{-1}\ \mathrm{W/m^2}$$

(f)

$$\frac{1}{2}|\vec{\boldsymbol{S}}| = \frac{27}{4\pi} \times 10^{-1}\ \mathrm{W}$$

(g)

0